On Constitutive Equations For Elastomers And Elastomeric Foams

William W. Feng
John O. Hallquist
Livermore Software Technology Corporation
7374 Las Positas Road
Livermore, CA 94551

Abstract

The Hill-Ogden elastic constitutive equation for incompressible and compressible rubber-like materials is presented. The derivation and computer programs to determine the material constants for these equations from uniaxial and biaxial tests are included. These constitutive equations and the computer programs for determining the material constants have been implemented into LS-DYNA. A few examples are shown.

Some special cases are given to demonstrate the versatility of these constitutive equations. The Mooney-Rivlin constitutive equation is a special case. The Feng-Christensen viscoelastic foam model in one-dimensional compression, developed in 1986, can be written in a mathematical form and implemented in finite element codes.

Introduction

Rubber or rubber-like materials have been increasingly used in engineering design. Their designs require more numerical calculations. Yet to this date the development of constitutive equations has not kept up with this demand. We commonly use neo-Hookean and Mooney-Rivlin constitutive equations to model rubber behavior. These are incompressible-elastic constitutive equations. Yet rubber, and in particular elastomeric foams, are compressible and viscoelastic. The experimental phenomena are shown in Figures 1, 2 and 3. Figure 1 shows cylindrical foam being compressed. Figure 2 shows the stress-strain relationship. Figure 3 shows the relaxation phenomenon when the compressed cylinder is held at a constant deformation and the time history of stress is plotted.

In this paper the Hill-Ogden [1, 2] strain-energy density equation for highly compressible materials is introduced. These constitutive equations are further extended to highly compressible viscoelastic materials. Furthermore, experiments as well as numerical analysis for determining the material constants are mentioned here.

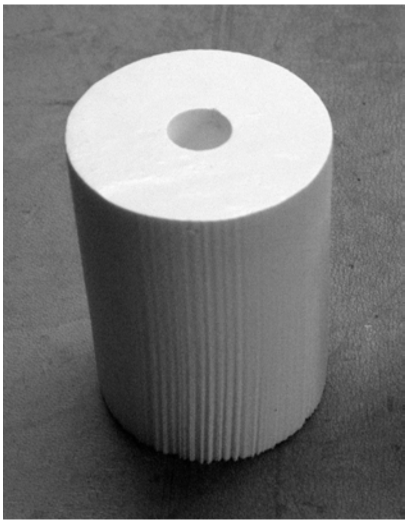

Fig. 1, A compression test specimen

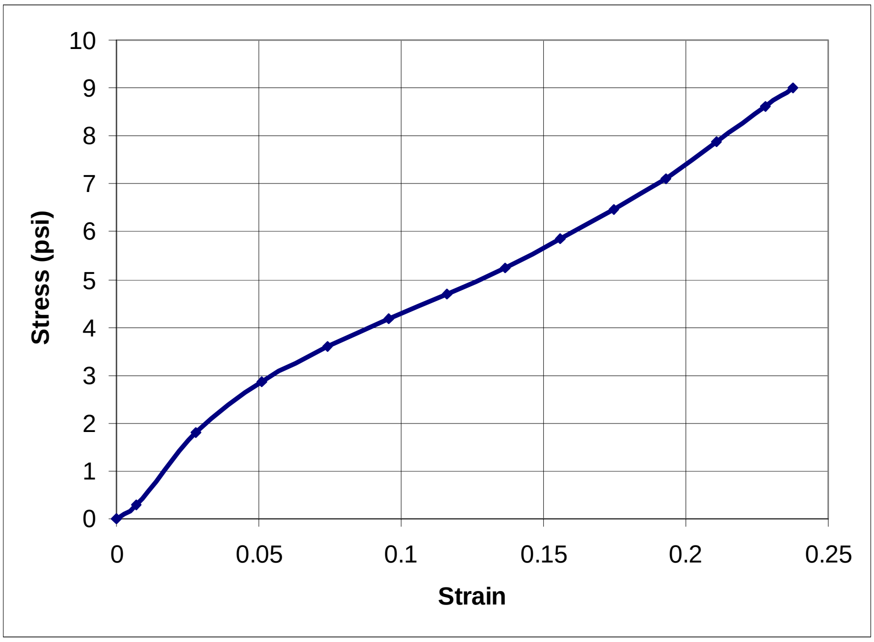

Fig. 2, A typical stress-strain curve for a foam

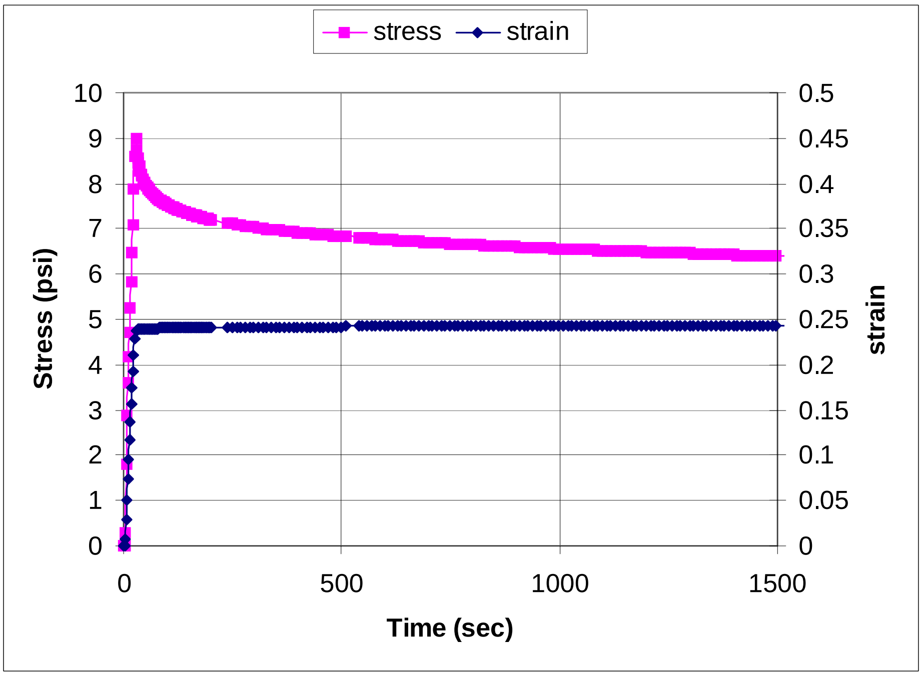

Fig. 3 A typical relaxation curve for a foam

The new constitutive equations are very general. For example, Mooney-Rivlin [3] and many currently used constitutive equations for rubber-like material are special cases. In 1986, Feng and Christensen [4, 5] developed a model to study elastomeric foams. Their model accurately described the behavior of foam; yet, no mathematical representation was attempted. Therefore, it could not be implemented into finite element codes. As a special case, in this paper, the Feng-Christensen foam model is written in a mathematical form and implemented in LS-DYNA [6].

Highly compressible elastic material

For highly compressible materials the Hill-Ogden [1, 2] strain-energy density equation can be written:

$$W = \sum_{j=1}^{m} \frac{C_j}{b_j} \left(\lambda_1^{b_j} + \lambda_2^{b_j} + \lambda_3^{b_j} - 3 + \frac{1}{n}\left(J^{-nb_j} - 1\right) \right) \qquad (1)$$

where C_j and b_j are material constants and $J = \lambda_1 \lambda_2 \lambda_3$ represents the ratio of deformed to undeformed volume. n is a compressibility material constant.

When n is a large number, the material is incompressible. When n is a small number, the material is highly compressible, like foam. The principal Cauchy stresses are

$$Jt_i = \sum_{j=1}^{m} C_j \left[\lambda_i^{b_j} - J^{-nb_j}\right],\ i = 1,2,3. \tag{2}$$

The nominal stresses (force per unit undeformed area) are

$$S_i = \frac{1}{\lambda_i} \sum_{j=1}^{m} C_j \left[\lambda_i^{b_j} - J^{-nb_j}\right],\ i = 1,2,3 \tag{3}$$

For hydrostatic state $t_1 = t_2 = t_3 = P$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the pressure-deformation relationship is

$$P = \sum_{j=1}^{m} C_j \left[\lambda^{b_j - 3} - \lambda^{-3(nb_j + 1)}\right] \tag{4}$$

From the above equations, the shear modulus for infinitesimal strain is

$$\mu = \frac{1}{2} \sum_{j=1}^{m} C_j b_j \cdot \tag{5}$$

The bulk modulus for infinitesimal strain is

$$\kappa = 2\mu\left(n + \frac{1}{3}\right). \tag{6}$$

The material constant n as well as C_j and b_j can be determined from the experimental data. For uniaxial tension or compression tests, $S_2 = S_3 = 0$, and $\lambda_2 = \lambda_3$, we have

$$\lambda_3 = \lambda_1^{-n/(2n+1)}. \tag{7}$$

The uniaxial-Cauchy stress, t_1, and stretch ratio, λ_1, relationship is

$$t_1 = \frac{1}{J}\sum_{j=1}^{m} C_j \left[\lambda_1^{b_j} - J^{-nb_j}\right] \quad (8)$$

and

$$J = \lambda_1^{\frac{1}{2n+1}} \quad (9)$$

The nominal uniaxial stress, S_1, is related to the stretch ratio, λ_1, by

$$S_1 = \sum_{j=1}^{m} C_j \left[\lambda_1^{b_j-1} - \lambda_1^{\frac{-nb_j}{2n+1}-1}\right] \quad (10)$$

Likewise, for equibiaxial tension or compression cases, $S_3 = 0$, $S_1 = S_2 = S$, and $\lambda_1 = \lambda_2 = \lambda$, we have

$$\lambda_3 = \lambda^{\frac{-2n}{n+1}}. \quad (11)$$

The Cauchy equibiaxial stress, *t*, is related to the stretch ratio, λ, by

$$t = \frac{1}{J}\sum_{j=1}^{m} C_j \left[\lambda^{b_j} - J^{-nb_j}\right] \quad (12)$$

and

$$J = \lambda^{\frac{2}{n+1}} \quad (13)$$

The nominal equibiaxial stress S, is then related to the stretch ratio, λ, by

$$S = \sum_{j=1}^{m} C_j \left[\lambda^{b_j-1} - \lambda^{\frac{-2nb_j}{n+1}-1}\right] \quad (14)$$

Hence, the constant n can be easily determined from the relationships between stretch ratios; equation (7) for uniaxial tests and equation (11) for equibiaxial tests. The material constants C_j and b_j can be determined from either equation (10) for uniaxial tests or equation (14) for equibiaxial tests.

The material type 177 Ogden foam and the subroutines for determining the material constants in LS-DYNA are based on these equations.

When $C_1 = 200$, $C_2 = -20$, $b_1 = 2$, $b_2 = -2$ and $n = \infty$, the results obtained from material type 177 is the same as the results obtained from material type 27, the Mooney-Rivlin material when $A =$ 100, and $B =$ 10. For a numerical test, the four undeformed and deformed solid elements are shown in Figures 4 and 5. Four uniaxial-compressive-Cauchy-stress states, one obtained from material type 27, and three obtained from material type 177, are shown in Figure 6. Element No. 1 is obtained from material type 27, and elements No. 2-4 are from material type 177. The effects of n are shown. For element No. 2, $n = 20$, a nearly incompressible material, the result is the same as for element No. 1. For element No. 3, $n = 2$, for element No. 4, $n = 0.2$. The time scale shown in Figure 6, *t* = 0.0, 0.01, 0.02, 0.03, and 0.04, correspond to λ_1 = 1.0, 0.8, 0.6, 0.4 and 0.2. Between *t* = 0.04 and 0.05, λ_1 remains 0.2.

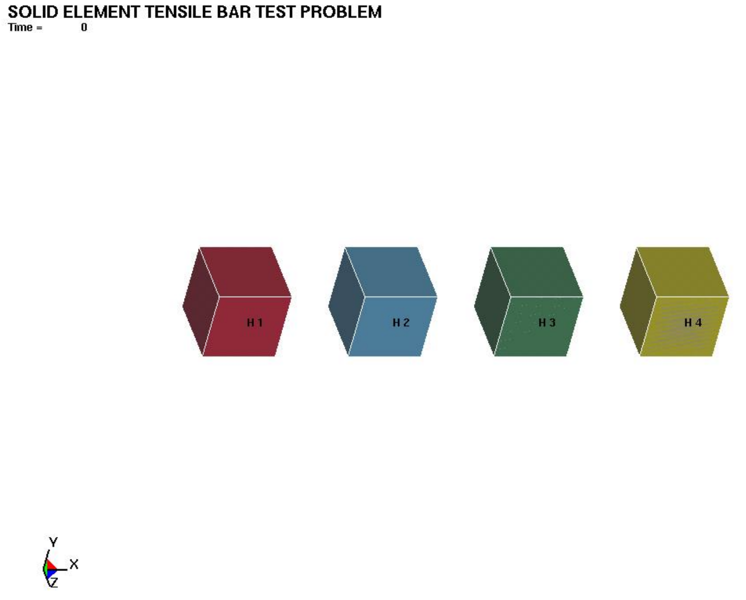

Fig. 4 Four undeformed solid elements

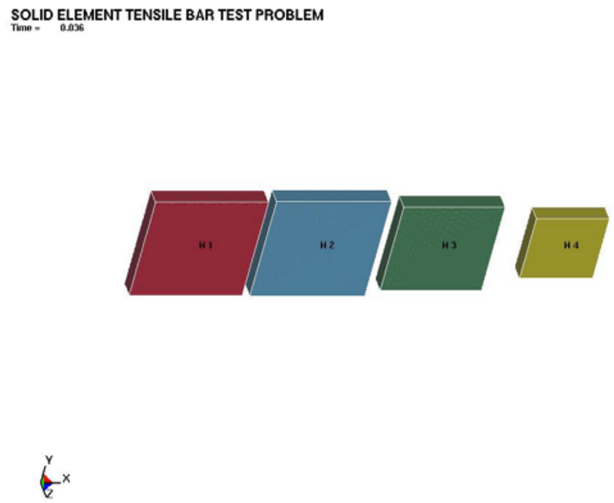

Fig. 5 Four deformed solid elements

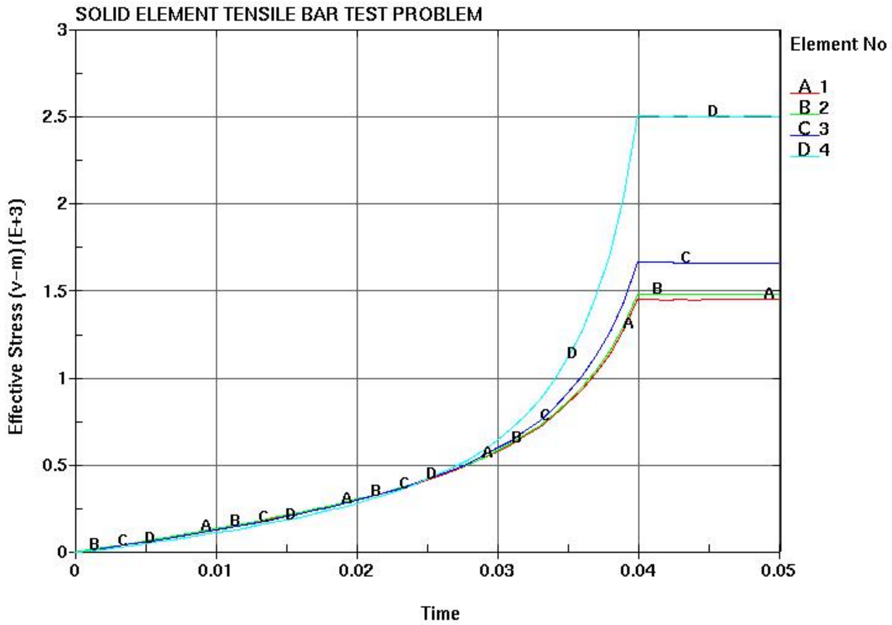

Fig. 6 Uniaxial-compressive-Cauchy-stress (effective stress shown) from LS-DYNA

When $C_1 = 200$, $C_2 = 20$, $b_1 = 2$, $b_2 = -2$ and n = 200, 2, 0.2 and 0.0002, the uniaxial-compressive-Cauchy-stresses obtained from equation (8) are shown in Figure 7. The LS-DYNA results are the same.

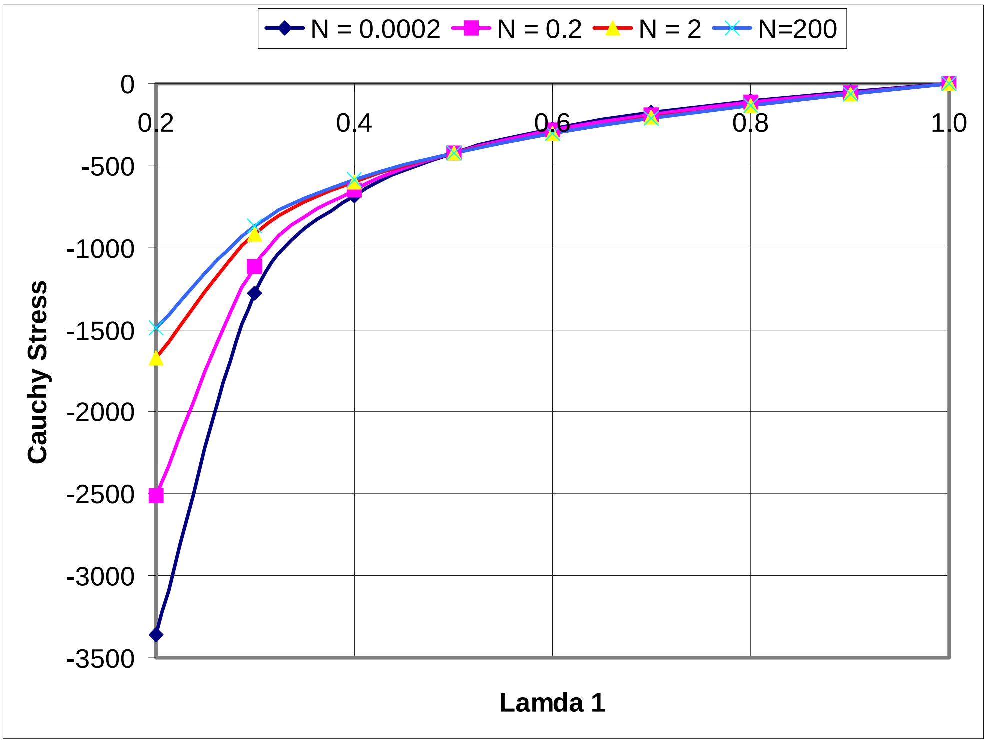

Fig. 7 Uniaxial-compressive-Cauchy-stress obtained from Eq. (8)

The uniaxial-compressive-nominal stress is given in equation (10). When $C_1 = 200$, $C_2 = 20$, $b_1 = 2$, $b_2 = -2$ and n = 200, 2, 0.2 and 0.0002, the uniaxial-compressive-nominal stresses obtained from equation (10) are shown in Figure 8.

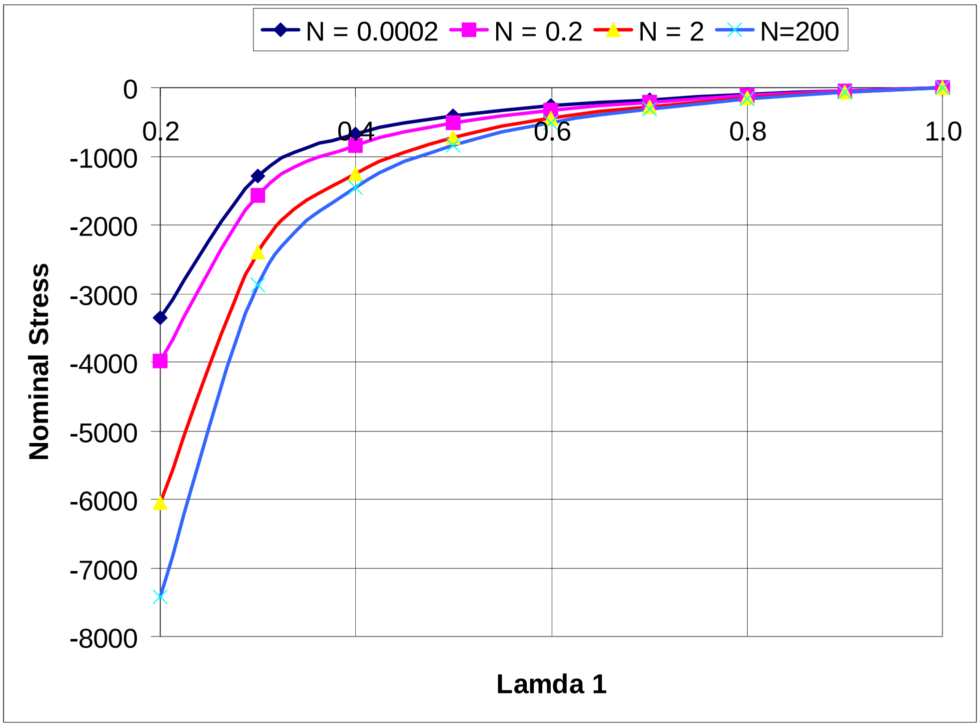

Fig. 8 Uniaxial-compressive-nominal-stress obtained from Eq. (10)

Viscoelastic effect

The viscoelastic effects are taken into account through linear viscoelasticity by a convolution integral of the form:

$$\sigma_{ij} = \int_0^t g_{ijkl}(t-\tau)\frac{\partial \varepsilon_{kl}}{\partial \tau}d\tau \tag{15}$$

or in terms of the second Piola-Kirchhoff stress, S_{ij}, and Green's strain tensor, E_{ij}.

$$S_{ij} = \int_0^t G_{ijkl}(t-\tau)\frac{\partial E_{kl}}{\partial \tau}d\tau \tag{16}$$

where $g_{ijkl}(t)$ and $G_{ijkl}(t)$ are the relaxation functions for the different stress measures. These stress components are added to the stress tensor determined from the strain energy functional.

The relaxation function is represented by the Prony series:

$$g(t) = \sum_{i=1}^{N} G_i e^{-\gamma_i t} \tag{17}$$

The shear moduli are G_i, and decay constants are γ_i.

The material type 178, viscoelastic-Hill foam, in LS-DYNA is based on these equations coupled with the constitutive equation for highly compressible elastic materials.

For a numerical test, the four undeformed and deformed solid elements are shown in Figures 4 and 9 respectively. The elastic material constants used in the calculation are the same as used in the previous section, i.e., $C_1 = 200$, $C_2 = -20$, $b_1 = 2$, $b_2 = -2$. The shear moduli and decay constants are

i	G_i	γ_i
1	1.0000E+02	0.1000E+00
2	1.0000E+02	0.1000E+01
3	1.0000E+02	0.1000E+02
4	1.0000E+02	0.1000E+03

(18)

Four uniaxial-compressive-Cauchy-stress states, two obtained from material type 177 and two obtained from material type 178, are shown in Figure 10. Element Nos. 1 and 2 are obtained from material type 177 for elastic materials. Element Nos. 3 and 4 are from material type 178 for viscoelastic materials. Element Nos. 1 and 3, $n = 20$, are nearly incompressible elements. Element Nos. 2 and 4, $n = 0.0$, are highly compressible elements. The linear ramp displacement is applied between t = 0.0, λ_1 = 1.0 and t = 0.01, λ_1 = 0.2. Between t = 0.01 and 0.05, λ_1 = 0.2, a constant.

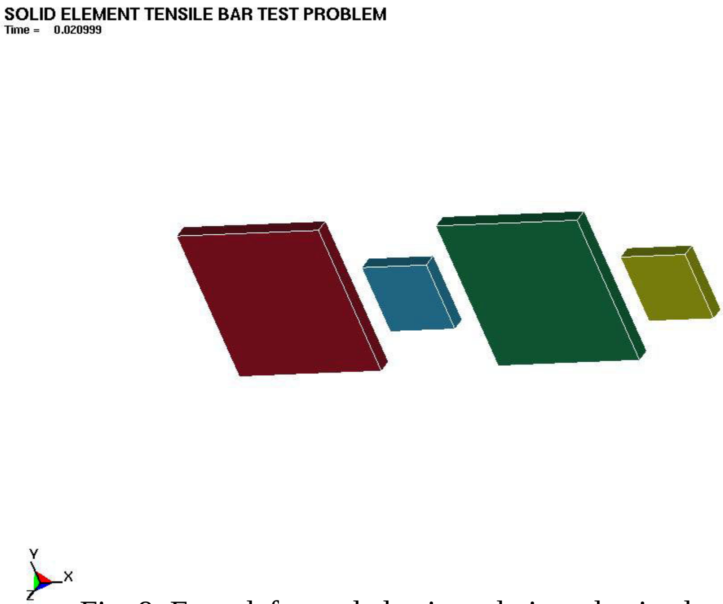

Fig. 9 Four deformed elastic and viscoelastic elements

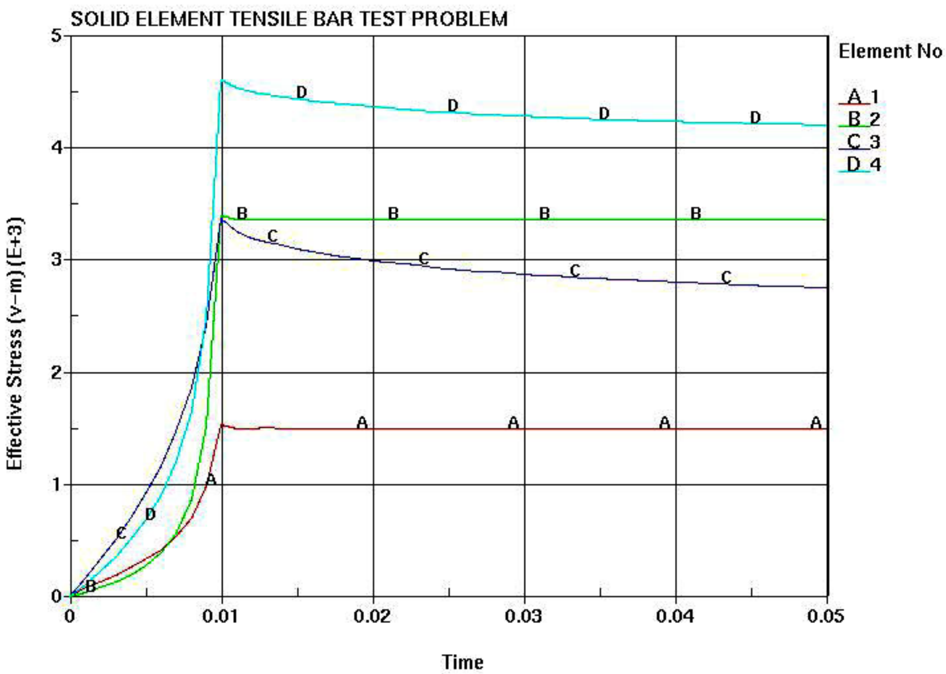

Fig. 10 Uniaxial-compressive-Cauchy-stress (effective stress shown) from LS-DYNA for the deformed elastic and viscoelastic solid elements

Feng-Christensen foam model

The Feng-Christensen foam model [4, 5] is a concentric hollow sphere. It assumes no lateral deformation during compression. The model describes well most foam during compression; however, there are no mathematical equations presented in their paper. Therefore, it could not be used in the finite element codes. The Feng-Christensen foam model in one-dimensional compression is a special case of the constitutive equation presented in this paper, i.e., $\lambda_2 = 1$; hence, $n = 0$. It is now implemented in LS-DYNA.

Determination of material constants

As mentioned in the above section, the constant n can be easily determined from the relationships between stretch ratios test data. The material constants C_j and b_j can be determined from relationships between the nominal stress and stretch ratio.

Subroutines for determining these constants from uniaxial or equibiaxial experimental data were implemented into LS-DYNA. In order to check the accuracy of LS-DYNA, the test data were obtained by a numerical simulated experiment. These test data were generated by Excel. The input values, m, n, C_j and b_j, for Excel are:

$$
\begin{array}{ll}
m = 2 & n = 0.21, \\
C_1 = 310.0, & b_1 = 2.0, \\
C_2 = -31.0, & b_2 = -2.0.
\end{array}
\tag{19}
$$

LS-DYNA determines the material constants from the simulated experimental data. On the basis of the simulated uniaxial experimental data, with $m = 4$ and the range of b = -8 to 8, the determined elastic material constants are:

$$
\begin{array}{ll}
n = 0.21 & \\
C_1 = 0.3176E+00, & b_1 = -0.4016E+01, \\
C_2 = -0.3108E+02, & b_2 = -0.2032E+01, \\
C_3 = 0.3101E+03, & b_3 = 0.2000E+01, \\
C_4 = 0.7732E-03, & b_4 = 0.3520E+01.
\end{array}
\tag{20}
$$

The results are almost exact. However, due to the mathematical nature of the constitutive equations (2 and 3), there are many sets of roots for C_j and

b_j. The users must know the limitations of the mathematical form and pick the constants that are physically sound.

For viscoelastic material, constants C_j, b_j and n can be determined from long-term test data ($t = \infty$). The method is the same as given in the above. The material constants for the relaxation $g(t)$ can be determined from relaxation test data.

Conclusions

The constitutive equations for highly compressible elastic and viscoelastic materials have been presented in this paper. These constitutive equations have been implemented in LS-DYNA. Some check problems were also presented. These results should have direct application to rubber-like materials and to elastomeric foams. The input values for LS-DYNA will either be the material constants or data from experimental tests. Methods for determining elastic and viscoelastic constants are also presented.

References

[1] Hill, R., Adv. Appl. Mech. 18, 1, (1978.)

[2] Ogden, R. W., “Recent Advances in the Phenomenological Theory of Rubber Elasticity,” Rubber Chemistry and Technology, Vol. 59, No 3, P. 386, (1986.)

[3] Mooney, M., J. Appl. Phys. 11, p. 582, (1940.)

[4] Feng, W. W., and Christensen, R. M., "Nonlinear Deformation of Elastomeric Foams," International J. of Non-Linear Mechanics, Vol. 17, p. 355, (1982.)

[5] Christensen, R. M., and W. W. Feng, "Nonlinear Compressive Deformation of Viscoelastic Porous Materials," The J. of Mechanics and Materials, Vol. 2, p. 239, (1983.)

[6] Hallquist, J. O., “LS-DYNA Version 970,” Livermore Software Technology Corporation, to be released.